

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

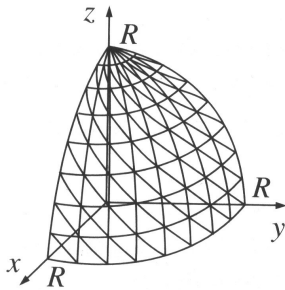
$$\nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \quad \text{and} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

1. (8%, 12%) $\mathbf{v} = r^2 \cos \theta \hat{\mathbf{r}} + r^2 \cos \phi \hat{\boldsymbol{\theta}} - r^2 \cos \theta \sin \phi \hat{\boldsymbol{\phi}}$

(a) Compute $\nabla \cdot \mathbf{v}$.

(b) Check the divergence theorem using the volume shown in the figure (one octant of the sphere of radius R).

[Hint: Make sure you include the entire surface.]

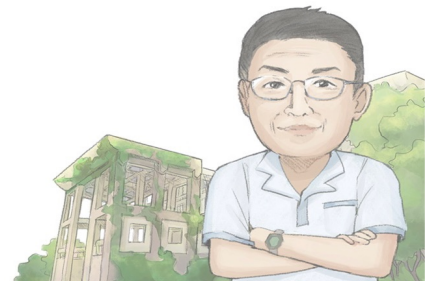
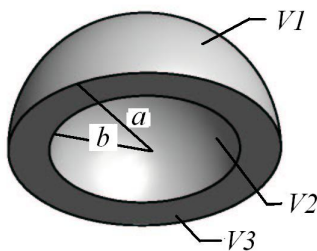


2. (10%, 10%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a, \theta) = 0$, $V_2(b, \theta) = V_0(2 \cos \theta - 5 \cos \theta \sin^2 \theta)$, $V_3(r, \pi/2) = 0$. V_0 is a constant.

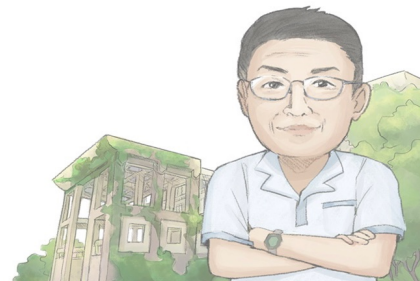
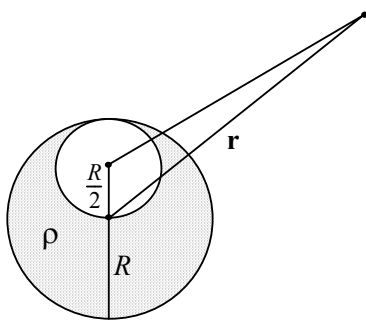
(a) Show the general solution in the region $b \leq r \leq a$ and determine the potential in the region $b \leq r \leq a$, using the boundary conditions.

(b) When $V_2(b, \theta) = V_0 \sin \theta$ and $V_1(a, \theta) = V_3(r, \pi/2) = 0$, how do you solve this problem? Please explain as detailed as possible.

[Hint: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, and $P_3(x) = (5x^3 - 3x)/2$.]



3. (7%, 7%, 6%) The potential of some configuration is given by the expression $V(\mathbf{r}) = A e^{-\lambda r} / r$, where A and λ are constants.
- Find the energy density (energy per unit volume).
 - Find the charge density $\rho(\mathbf{r})$.
 - Find the total charge Q (do it two different ways) and verify the divergence theorem.
4. (7%,7%,6%) A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane.
- Find the potential V in the region above the plane.
 - Find the surface charge density σ induced on the conducting plane.
 - Find the force on the wire per unit length.
- [Hint: Use the method of images.]
5. (8%, 6%, 6%) Consider a hollowed charged sphere with radius R and uniform charge density ρ as shown in the figure. The inner radius of the spherical cavity is $R/2$.
- If the observer is very far from the charged sphere, find the multiple expansion of the potential V in power of $1/r$
 - Find the dipole moment $\mathbf{p}$.
 - Find the electric field $\mathbf{E}$ up to the dipole term.
- [Note: Specify a vector with both magnitude and direction.]



1.

(a)

$$\mathbf{v} = r^2 \cos \theta \hat{\mathbf{r}} + r^2 \cos \phi \hat{\boldsymbol{\theta}} - r^2 \cos \theta \sin \phi \hat{\boldsymbol{\phi}}$$

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta r^2 \cos \phi) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (-r^2 \cos \theta \sin \phi) \\ &= 4r \cos \theta + r \frac{\cos \theta}{\sin \theta} \cos \phi - r \frac{\cos \theta}{\sin \theta} \cos \phi \\ &= 4r \cos \theta \end{aligned}$$

(b)

The divergence theorem $\int_V \nabla \cdot \mathbf{v} d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$

$$\begin{aligned} \int_V \nabla \cdot \mathbf{v} d\tau &= \int_0^R \int_0^{\pi/2} \int_0^{\pi/2} (4r \cos \theta) r^2 \sin \theta dr d\theta d\phi = \frac{\pi}{2} \int_0^R \int_0^{\pi/2} (4r^3 \cos \theta \sin \theta) dr d\theta \\ &= \frac{\pi}{16} \int_0^R \int_0^{\pi/2} (4 \sin 2\theta) dr^4 d\theta = \frac{\pi}{4} R^4 \int_0^{\pi/2} \sin 2\theta d2\theta = \frac{\pi}{4} R^4 \end{aligned}$$

$\oint_S \mathbf{v} \cdot d\mathbf{a}$ = xy-plane + yz-plane + zx-plane + curved surface

xy-plane: $d\mathbf{a} = -r dr d\theta \hat{\boldsymbol{\phi}}$, $\phi = 0$, $\mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \theta \sin \phi) r dr d\theta = 0$,

yz-plane: $d\mathbf{a} = r dr d\theta \hat{\boldsymbol{\phi}}$, $\phi = \pi/2$, $\mathbf{v} \cdot d\mathbf{a} = -(r^2 \cos \theta \sin \phi) r dr d\theta = -r^3 \cos \theta dr d\theta$, $\int_0^R \int_0^{\pi/2} -(r^3 \cos \theta) dr d\theta = -\frac{1}{4} R^4$

zx-plane: $d\mathbf{a} = r dr d\phi \hat{\boldsymbol{\theta}}$, $\theta = \pi/2$, $\mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \phi) r dr d\phi = r^3 \cos \phi dr d\phi$, $\int_0^R \int_0^{\pi/2} (r^3 \cos \phi) dr d\phi = \frac{1}{4} R^4$

curved surface: $d\mathbf{a} = R^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}$, $r = R$, $\mathbf{v} \cdot d\mathbf{a} = (R^2 \cos \theta) R^2 \sin \theta d\theta d\phi = \frac{R^4}{2} \sin 2\theta d\theta d\phi$,

$$\int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{R^4}{2} \sin 2\theta \right) d\theta d\phi = \frac{\pi}{4} R^4$$

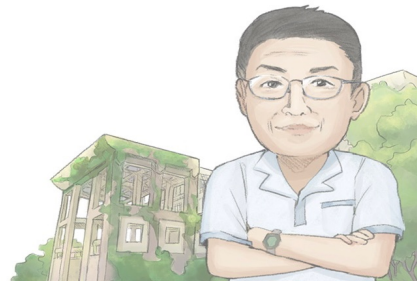
$$\oint_S \mathbf{v} \cdot d\mathbf{a} = 0 - \frac{1}{4} R^4 + \frac{1}{4} R^4 + \frac{\pi}{4} R^4 = \frac{\pi}{4} R^4 = \int_V \nabla \cdot \mathbf{v} d\tau$$

2.

(a)

$$\text{Boundary condition} \begin{cases} \text{(i)} V_1(a, \theta) = 0 \\ \text{(ii)} V_2(b, \theta) = V_0(2 \cos \theta - 5 \cos \theta \sin^2 \theta) = V_0(5 \cos^3 \theta - 3 \cos \theta) = 2V_0 P_3 \\ \text{(ii)} V_3(r, \theta = \pi/2) = 0 \end{cases}$$

$$\text{General solution } V(r, \theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos \theta)$$



$$\text{B.C. (i)} \rightarrow V(a, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) P_{\ell}(\cos \theta) = 0 \Rightarrow B_{\ell} = -A_{\ell} a^{2\ell+1}$$

$$\text{B.C. (ii)} \rightarrow V(b, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} b^{\ell} + B_{\ell} b^{-(\ell+1)}) P_{\ell}(\cos \theta) = 2V_0 P_3(\cos \theta)$$

Comparing the coefficient $\Rightarrow A_3 b^3 + B_3 b^{-4} = 2V_0$, $A_{\ell} = B_{\ell} = 0$ for $\ell = 0, 1, 2, 4, 5, \dots$

$$\text{B.C. (iii)} \rightarrow V(r, \theta = \frac{\pi}{2}) = (A_3 r^3 + B_3 r^{-4}) P_3(0) = 0$$

$\Rightarrow A_{\ell} = B_{\ell} = 0$ except $\ell = 3$,

$$A_3 = \frac{2V_0 b^4}{b^7 - a^7} \text{ and } B_3 = -\frac{2V_0 b^4 a^7}{b^7 - a^7}$$

$$\therefore V(r, \theta) = \left(\frac{2V_0}{b^7 - a^7} b^4 r^3 - \frac{2V_0}{b^7 - a^7} b^4 a^7 r^{-4} \right) \left(\frac{5 \cos^3 \theta - 3 \cos \theta}{2} \right)$$

(b)

$$\text{Boundary condition} \begin{cases} \text{(i)} V_1(a, \theta) = 0 \\ \text{(ii)} V_2(b, \theta) = V_0 \sin \theta \\ \text{(iii)} V_3(r, \theta = \pi/2) = 0 \end{cases}$$

$$\text{General solution } V(r, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(\cos \theta)$$

$$\text{B.C. (i)} \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) P_{\ell}(\cos \theta) = 0 \Rightarrow B_{\ell} = -A_{\ell} a^{2\ell+1}$$

$$\text{B.C. (iii)} \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(0) = 0 \Rightarrow \ell=1, 3, 5, \dots \text{ only odd terms survive}$$

$$\text{B.C. (ii)} \sum_{\ell=0}^{\infty} A_{\ell} (b^{\ell} - \frac{a^{2\ell+1}}{b^{\ell+1}}) P_{\ell}(\cos \theta) = V_0 \sin \theta$$

$$\int_{-1}^1 P_{\ell}(x) P_{\ell'}(x) dx = \int_0^{\pi} P_{\ell}(\cos \theta) P_{\ell'}(\cos \theta) \sin \theta d\theta = \begin{cases} 0 & \text{if } \ell' \neq \ell \\ \frac{2}{2\ell+1}, & \text{if } \ell' = \ell \end{cases}$$

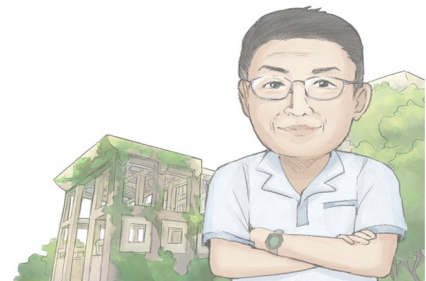
$$\sum_{\ell=0}^{\infty} A_{\ell} (b^{\ell} - \frac{a^{2\ell+1}}{b^{\ell+1}}) \int_0^{\pi} P_{\ell}(\cos \theta) P_{\ell'}(\cos \theta) \sin \theta d\theta = \int_0^{\pi} V_0 \sin \theta P_{\ell'}(\cos \theta) \sin \theta d\theta$$

$$A_{\ell} = \left(\frac{b^{\ell+1}}{b^{2\ell+1} - a^{2\ell+1}} \right) \frac{2\ell+1}{2} \int_0^{\pi} V_0 \sin \theta P_{\ell}(\cos \theta) \sin \theta d\theta$$

But $A_{\ell} = 0$ for $\ell=1, 3, 5, \dots$ It does not make sense. Why?

$$\text{Add an artificial boundary condition } V_2(b, \theta) = \begin{cases} V_0 \sin \theta & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ -V_0 \sin \theta & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

$$\text{or } V_2(b, \theta) = \begin{cases} V_0 \sin \theta & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$



3.

(a)

$$\mathbf{E} = -\nabla V = -A \frac{\partial}{\partial r} \left(\frac{e^{-\lambda r}}{r} \right) \hat{\mathbf{r}} = -A \left\{ \frac{-\lambda r e^{-\lambda r} - e^{-\lambda r}}{r^2} \right\} \hat{\mathbf{r}} = A \frac{(\lambda r + 1)e^{-\lambda r}}{r^2} \hat{\mathbf{r}}$$

$$\text{Energy density} = \frac{\epsilon_0}{2} E^2 = \frac{\epsilon_0}{2} A^2 \frac{(\lambda r + 1)^2 e^{-2\lambda r}}{r^4}$$

(b)

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 A \left(\nabla \cdot \frac{(\lambda r + 1)e^{-\lambda r}}{r^2} \hat{\mathbf{r}} \right) = \epsilon_0 A (\lambda r + 1) e^{-\lambda r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) + \epsilon_0 A \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla ((\lambda r + 1)e^{-\lambda r})$$

$$\left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r}) \quad \text{and} \quad (\lambda r + 1)e^{-\lambda r} \delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$$

$$\frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla ((\lambda r + 1)e^{-\lambda r}) = \frac{\hat{\mathbf{r}}}{r^2} \cdot \hat{\mathbf{r}} \frac{\partial}{\partial r} ((\lambda r + 1)e^{-\lambda r}) = \frac{1}{r^2} \frac{\partial}{\partial r} ((\lambda r + 1)e^{-\lambda r}) = -\frac{\lambda^2}{r} e^{-\lambda r}$$

$$\rho = \epsilon_0 A \left[4\pi \delta^3(\mathbf{r}) - \frac{\lambda^2}{r} e^{-\lambda r} \right]$$

(c)

$$Q = \int_V \rho d\tau = \int_V \epsilon_0 A \left[4\pi \delta^3(r) - \frac{\lambda^2}{r} e^{-\lambda r} \right] d\tau = 4\pi \epsilon_0 A \left[1 + \int_{r=0}^{\infty} \frac{\lambda^2}{r} e^{-\lambda r} r^2 dr \right]$$

$$\int_{r=0}^{\infty} \frac{\lambda^2}{r} e^{-\lambda r} r^2 dr = \int_{r=0}^{\infty} e^{-\lambda r} \lambda^2 r dr = - \int_{r=0}^{\infty} \lambda r de^{-\lambda r} = - \int_{x=0}^{\infty} x de^{-x} = -1 \Rightarrow Q = \int_V \rho d\tau = 0$$

Use Gauss's law, the charge enclosed in a sphere of radius R

$$Q_R = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{a} = 4\pi \epsilon_0 A (\lambda R + 1) e^{-\lambda R} \Rightarrow \text{The total charge } Q_{R \rightarrow \infty} = 4\pi \epsilon_0 A (\lambda R + 1) e^{-\lambda R} \Big|_{R=\infty} = 0$$

4.

(a)

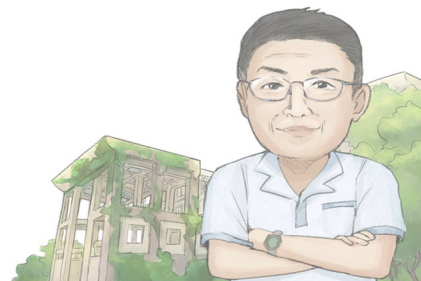
Assume the image line charge of $-\lambda$ is placed at a distance d below the plane.

Using the Gauss's law, the electric field outside a line charge λ is $\mathbf{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{\mathbf{r}}$.

$$\text{So } V = - \int_{r_0}^r \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r} = V(r) - V_{ref}(r_0)$$

$$V = V_+ + V_- = \frac{\lambda}{2\pi\epsilon_0} \left(\ln \frac{r_0}{\sqrt{(x-d)^2 + y^2}} - \ln \frac{r_0}{\sqrt{(x+d)^2 + y^2}} \right) = \frac{\lambda}{4\pi\epsilon_0} \left(\ln \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2} \right)$$

(b)



$$\sigma = \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \epsilon_0 E_x = -\frac{\partial}{\partial x} \frac{\lambda}{4\pi} \left(\ln \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2} \right) \Bigg|_{x=0} = -\frac{\lambda}{4\pi} \left(\frac{2(x+d)}{(x+d)^2 + y^2} - \frac{2(x-d)}{(x-d)^2 + y^2} \right) \Bigg|_{x=0}$$

$$= -\frac{\lambda}{4\pi} \frac{4d}{d^2 + y^2} = -\frac{\lambda}{\pi} \frac{d}{d^2 + y^2}$$

Simple check: $\lambda' = \int_{-\infty}^{\infty} \sigma dy = \int_{-\infty}^{\infty} -\frac{\lambda}{\pi} \frac{d}{d^2 + y^2} dy$

Let $y = d \tan \theta$, $dy = d \sec^2 \theta d\theta$

$$\lambda' = -\frac{\lambda}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d^2 \sec^2 \theta}{d^2 \sec^2 \theta} d\theta = -\lambda$$

(c)

$$dF = Edq = E\lambda d\ell$$

$$\frac{dF}{d\ell} = E\lambda = \frac{\lambda}{2\pi\epsilon_0(2d)} \lambda = \frac{\lambda^2}{4\pi\epsilon_0 d}$$

5.

(a)

Consider this problem as two charge spheres, one with charge density ρ the other with opposite charge density $-\rho$.

$$V_{big} = \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} R^3 \right) \quad \text{and} \quad V_{small} = \frac{1}{4\pi\epsilon_0 \left| \mathbf{r} - \frac{1}{2} \mathbf{R} \right|} \left(-\rho \frac{4\pi}{3} \left(\frac{R}{2} \right)^3 \right)$$

$$\frac{1}{\left| \mathbf{r} - \frac{1}{2} \mathbf{R} \right|} = \frac{1}{r} \left(1 + \left(\frac{\frac{1}{2}R}{r} \right) \cos \theta + \dots \right)$$

Using the principle of superposition, we find,

$$V = \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} R^3 \right) - \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} \left(\frac{R}{2} \right)^3 \right) \left(1 + \left(\frac{\frac{1}{2}R}{r} \right) \cos \theta + \dots \right)$$

$$= \frac{1}{4\pi\epsilon_0 r} \frac{7}{8} \left(\rho \frac{4\pi}{3} R^3 \right) - \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} \left(\frac{R}{2} \right)^3 \right) \left(\frac{R}{2r} \right) \cos \theta + \dots, \quad \text{let } Q = \rho \frac{4\pi}{3} R^3$$

$$= \frac{1}{4\pi\epsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\epsilon_0 r^2} \left(\frac{Q}{8} \frac{R}{2} \right) \cos \theta + \dots$$

(b)

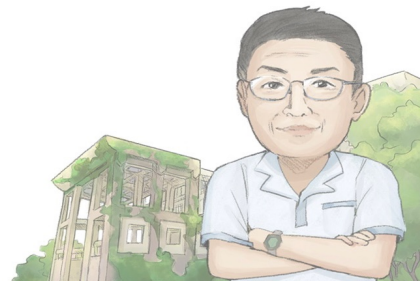
$$Q = \rho \frac{4\pi}{3} R^3$$

$$V = \frac{1}{4\pi\epsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\epsilon_0 r^2} \left(\frac{Q}{8} \frac{R}{2} \right) \cos \theta + \dots$$

The first term is the monopole term and the second term is the dipole term.

So the dipole moment $\mathbf{p} = -\frac{QR}{16} \hat{\mathbf{z}}$.

(c)



$$V = \frac{1}{4\pi\epsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\epsilon_0 r^2} \left(\frac{Q}{8} \frac{R}{2} \right) \cos\theta + \dots$$

$$\begin{aligned} \mathbf{E} = -\nabla V &= -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= \left(\frac{1}{4\pi\epsilon_0 r^2} \frac{7Q}{8} - \frac{2p}{4\pi\epsilon_0 r^3} \cos\theta \right) \hat{\mathbf{r}} - \frac{p}{4\pi\epsilon_0 r^3} \sin\theta \hat{\boldsymbol{\theta}} \end{aligned}$$

